# G-LEVELS OF PERFECT COMPLEXES AND A BASS FORMULA FOR LEVELS

LARS WINTHER CHRISTENSEN, ANTONIA KEKKOU, JUSTIN LYLE, ZACHARY NASON, AND ANDREW J. SOTO LEVINS

ABSTRACT. We prove that a commutative noetherian ring $R$ is Gorenstein of dimension at most $d$ if $d+1$ is an upper bound on the G-levels of perfect $R$-complexes. For $R$ local, we prove a formula for levels, with respect to injective or Gorenstein injective $R$-modules, of $R$-complexes with finitely generated homology; it mimics Bass' classic formula for injective dimension of finitely generated $R$-modules.

## 1. INTRODUCTION

Throughout this paper, $R$ denotes a commutative noetherian ring. Let $\mathsf{C}$ be a collection of objects in $\mathsf{D_b}(R)$, the bounded derived category over $R$. The number of mapping cones needed, up to summands, finite direct sums, and shifts, to build an $R$-complex $M$ from $\mathsf{C}$ is known as the level of $M$ with respect to $\mathsf{C}$ and denoted by $\operatorname{level}_R^{\mathsf{C}} M$. That is, levels stratify the derived category of $R$ using its triangulated structure. Levels with respect to the class $\{R\}$, sensibly known as $R$-levels, have been studied extensively. From the first study of $R$-levels by Avramov, Buchweitz, Iyengar, and Miller [3] it is known that $R$ is regular of Krull dimension $d$ if and only if every complex in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ has $R$-level at most $d+1$.

In this paper, we are concerned with levels with respect to the collection of finitely generated Gorenstein projective modules, denoted $\mathsf{G}(R)$, and the collections of Gorenstein injective and Gorenstein flat modules, denoted $\mathsf{GI}(R)$ and $\mathsf{GF}(R)$. A first major result in this area was obtained by Aihara, Araya, Iyama, Takahashi, and Yoshiwaki [1], who proved that if $R$ is Gorenstein of finite Krull dimension, then the $\mathsf{G}(R)$-levels of complexes in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ are bounded above by $\max\{2, \dim R + 1\}$. A converse was established by Awadalla and Marley [4], who proved that $R$ is Gorenstein of finite Krull dimension if there is an upper bound on the $\mathsf{G}(R)$-levels of complexes in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$.

Our first main result, Theorem 2.10, improves the result of Awadalla and Marley, as we show that $R$ is Gorenstein of finite Krull dimension if there is an upper bound on the $\mathsf{G}(R)$-levels of perfect $R$-complexes. This parallels a well-known characterization of regular rings; see for example Krause [11]. In Theorem 3.7 we combine Theorem 2.10 with results from [1] to give a comprehensive characterization of Gorenstein rings in terms of $\mathsf{G}(R)$-levels.

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Our second main result, Theorem 4.8, establishes an upper bound for the $\mathsf{GI}(R)$-levels of complexes in $\mathsf{D_b}(R)$. For $R$ local of positive depth, we obtain in Theorem 4.12 a formula for the $\mathsf{GI}(R)$-level of complexes in $\mathsf{D^f_b}(R)$ whose homology modules have finite Gorenstein injective dimension. This formula mimics Bass's classic formula for injective dimension of finitely generated $R$-modules.

We close by examining $\mathsf{GF}(R)$-levels of complexes in $\mathsf{D_b}(R)$. It follows from [1] that the $\mathsf{GF}(R)$-level of a complex $M$ in $\mathsf{D_b}(R)$ is at most $\max\{2, \operatorname{Gfd}_R \mathrm{H}(M)\}$, where $\mathrm{H}(M)$ is taken to be a module. We show that "max{2," is needed by constructing a complex of $\mathsf{GF}(R)$-level 2 whose homology modules are Gorenstein flat. As the underlying ring $R$ is regular, this is also a complex with flat homology modules whose level with respect to the class of flat $R$-modules is, nevertheless, 2.

## 2. G-Levels of perfect complexes

We start by introducing the most central notation; for any unexplained notation and terminology we refer the reader to Christensen, Foxby, and Holm [7].

For an $R$-complex

$$M \coloneqq \cdots \longrightarrow M_{i+1} \xrightarrow{\partial^M_{i+1}} M_i \xrightarrow{\partial^M_i} M_{i-1} \longrightarrow \cdots$$

and every integer $i$ set $\mathrm{Z}_i(M) \coloneqq \ker(\partial^M_i)$, $\mathrm{B}_i(M) \coloneqq \operatorname{im}(\partial^M_{i+1})$, $\mathrm{C}_i(M) \coloneqq \operatorname{coker}(\partial^M_{i+1})$, and $\mathrm{H}_i(M) \coloneqq \mathrm{Z}_i(M)/\mathrm{B}_i(M)$.

**2.1.** Let $\mathsf{C}$ be a collection of objects in the derived category $\mathsf{D}(R)$. Recall from [3] that the $\mathsf{C}$-level of a complex $M$ in $\mathsf{D}(R)$, denoted $\operatorname{level}^{\mathsf{C}}_R M$, is defined as:

(1) $\operatorname{level}^{\mathsf{C}}_R M = 0$ if $M$ is 0 in $\mathsf{D}(R)$.
(2) $\operatorname{level}^{\mathsf{C}}_R M = 1$ if $M$ is nonzero, but can be built from objects in $\mathsf{C}$ using shifts, summands, and finite direct sums.
(3) For $n > 1$, $\operatorname{level}^{\mathsf{C}}_R M = n$ if $n$ is the infimum of $i$ such that there is a triangle

$$K \longrightarrow L \oplus M \longrightarrow N \longrightarrow$$

with $\operatorname{level}^{\mathsf{C}}_R K = 1$ and $\operatorname{level}^{\mathsf{C}}_R N = i - 1$.

**2.2.** Let $\mathsf{C}$ be a collection of objects in $\mathsf{D}(R)$. A morphism $\alpha\colon M \to N$ in $\mathsf{D}(R)$ is $\mathsf{C}$-ghost if the induced maps

$$\operatorname{Ext}^n_R(C, M) \longrightarrow \operatorname{Ext}^n_R(C, N)$$

are zero for all integers $n$ and all objects $C$ in $\mathsf{C}$.

The next result was first proved by Kelly [10, Theorem 3].

**Ghost Lemma 2.3.** *Let $\mathsf{C}$ be a collection of objects from $\mathsf{D}(R)$ and $\alpha_i\colon M_i \to M_{i+1}$ for $0 \le i \le n-1$ a sequence of morphisms in $\mathsf{D}(R)$ such that $\alpha_{n-1}\alpha_{n-2}\cdots\alpha_0$ is a nonzero morphism in $\mathsf{D}(R)$. If each $\alpha_i$ is $\mathsf{C}$-ghost, then one has $\operatorname{level}^{\mathsf{C}}_R M_0 \ge n+1$.*

Let $\mathsf{G}(R)$ denote the collection of finitely generated Gorenstein projective $R$-modules.

**Lemma 2.4.** *Let $M$ be a finitely generated $R$-module. If there exists an integer $b \ge 1$ such that $\operatorname{Ext}^n_R(G, M) = 0$ holds for all $n \ge b$ and all $G$ in $\mathsf{G}(R)$, then $\operatorname{Ext}^n_R(G, M) = 0$ holds for all $n \ge 1$ and all $G$ in $\mathsf{G}(R)$.*

*Proof.* Assume that $\operatorname{Ext}_R^n(G,M)=0$ holds for all $n\geq b$ and all $G$ in $\mathsf{G}(R)$, and assume towards a contradiction that one has $\operatorname{Ext}_R^n(G,M)\neq 0$ for some $n\geq 1$ and some module $G$ from $\mathsf{G}(R)$. As $G$ is a $b^{\text{th}}$ syzygy of a module $G'$ in $\mathsf{G}(R)$, one has

$$\operatorname{Ext}_R^{b+n}(G',M)\cong \operatorname{Ext}_R^n(G,M)\neq 0\,,$$

a contradiction. □

Ideas for the proof of the next result come from [4, Theorem 3.3] and [11, Proposition A.1.2].

**Theorem 2.5.** *Let $R$ be a commutative noetherian local ring. If there exists an integer $\ell$ such that $\operatorname{level}_R^{\mathsf{G}(R)} P\leq \ell$ holds for every perfect $R$-complex $P$, then $R$ is Gorenstein.*

*Proof.* Without loss of generality one can assume that $\ell$ is at least 2. Let $k$ be the residue field of $R$, let $K$ be the Koszul complex on a minimal set of generators for the maximal ideal of $R$, and set $D \coloneqq \operatorname{Hom}_R(K,\operatorname{E}_R(k))$. Then $D$ is a bounded complex of injective $R$-modules and has finite length homology. To show that $R$ is Gorenstein, it is by [7, Corollary 19.5.12] enough to show that $D$ has finite projective dimension over $R$.

To this end, let $F\xrightarrow{\simeq} D$ be a semi-free resolution over $R$ with $F$ degreewise finitely generated. For $n\geq \sup \operatorname{H}(D)$, one has $\operatorname{pd}_R D\leq \operatorname{pd}_R \operatorname{C}_n(F)+n$, so it suffices to fix an integer $s\geq \sup\operatorname{H}(D)$ and show that $\operatorname{C}_s(F)$ has finite projective dimension; after a shift one can take $s=0$. As the injective dimension of $D$ over $R$ is finite, it follows that for every $R$-module $G$ one has $\operatorname{Ext}_R^m(G,D)=0$ for all $m>\operatorname{id}_R D$. Thus, for every $n\geq 0$ and $G$ in $\mathsf{G}(R)$, there is an integer $b_n$ such that $\operatorname{Ext}_R^m(G,\operatorname{C}_n(F))=\operatorname{Ext}_R^{m+n}(G,D)=0$ for all $m\geq b_n$, see Christensen, Frankild, and Holm [8, Corollary 2.10], whence one has $\operatorname{Ext}_R^m(G,\operatorname{C}_n(F))=0$ for all $m\geq 1$ and all $G\in\mathsf{G}(R)$ by Lemma 2.4.

For $n\geq 0$, set

$$X^n \coloneqq 0\longrightarrow F_{\ell+n}\xrightarrow{\partial_{\ell+n}}\cdots\xrightarrow{\partial_{n+1}} F_n\longrightarrow 0\,.$$

Notice that we have an exact sequence of $R$-complexes

$$0\longrightarrow \Sigma^{\ell+n}\operatorname{C}_{\ell+n+1}(F)\longrightarrow X^n\longrightarrow \Sigma^n\operatorname{C}_n(F)\longrightarrow 0\,,$$

and from the associated exact sequence of Ext modules one gets for every $G\in\mathsf{G}(R)$

$$(*)\qquad \operatorname{Ext}_R^m(G,X^n)=0 \quad\text{for}\quad m\neq -(\ell+n)\text{ and } m\neq -n\,.$$

Now, consider the canonical morphisms $\alpha^n\colon X^n\to X^{n+1}$ for $n=0,\dots,\ell-1$. It follows from $(*)$ that they are all $\mathsf{G}(R)$-ghost. Therefore, if the composite morphism $\alpha\coloneqq\alpha^{\ell-1}\alpha^{\ell-2}\cdots\alpha^0$ is nonzero in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$, then Ghost Lemma 2.3 yields

$$\operatorname{level}_R^{\mathsf{G}(R)} X^0\geq \ell+1\,,$$

which contradicts the assumption. Thus, $\alpha$ is a zero morphism in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$. Since $\alpha$ is a morphism of semi-projective $R$-complexes it is null-homotopic; in particular there exist homomorphisms

$$\sigma_\ell\colon F_\ell\longrightarrow F_{\ell+1}\quad\text{and}\quad \sigma_{\ell-1}\colon F_{\ell-1}\longrightarrow F_\ell$$

such that

$$1^{F_\ell}=\alpha_\ell=\sigma_{\ell-1}\partial_\ell+\partial_{\ell+1}\sigma_\ell\,.$$

Consider the composite

$$\mathrm{B}_\ell(F) \longrightarrow F_\ell \xrightarrow{\sigma_\ell} F_{\ell+1} \xrightarrow{\partial_{\ell+1}} \mathrm{B}_\ell(F)\,.$$

For $x$ in $\mathrm{B}_\ell(F)$, one has

$$x = 1^{F_\ell}(x) = \sigma_{\ell-1}\partial_\ell(x) + \partial_{\ell+1}\sigma_\ell(x) = \partial_{\ell+1}\sigma_\ell(x)\,,$$

so $\partial_{\ell+1}\sigma_\ell(x)$ is a left inverse to the inclusion $\mathrm{B}_\ell(F) \hookrightarrow F_\ell$, which means that $\mathrm{B}_\ell(F)$ is a summand of a free module, whence $\mathrm{pd}_R\,\mathrm{C}_0(F)$ is at most $\ell+1$. □

**Remark 2.6.** Notice that the complexes $X^n$ in the proof above have amplitude $\ell$. Thus, for $\ell \geq 2$ one gets the desired conclusion that $R$ is Gorenstein, as long as perfect $R$-complexes of amplitude $\ell$ have $\mathsf{G}(R)$-level at most $\ell$.

**2.7.** Recall that if $R$ is local and $M \in \mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ has finite Gorenstein dimension, then there is a classic formula due to Auslander and Bridger, see [7, Theorem 19.4.25]:

$$\mathrm{Gdim}_R\, M = \operatorname{depth} R - \operatorname{depth}_R M\,.$$

The following fact will be used several times in the next couple of proofs.

**2.8.** Let $\mathfrak{p}$ be a prime ideal in $R$ and $P$ a perfect $R_\mathfrak{p}$-complex. Since every finitely generated free $R_\mathfrak{p}$-module is a localization of a finitely generated free $R$-module, Letz's proof of [12, Lemma 3.9] shows that there exists a bounded complex $F$ of finitely generated free $R$-modules with $F_\mathfrak{p} \simeq P$ in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R_\mathfrak{p})$.

**Lemma 2.9.** *Let $R$ be Cohen–Macaulay of finite Krull dimension. There is a bounded complex $F$ of finitely generated free $R$-modules with* $\mathrm{level}_R^{\mathsf{G}(R)}\, F \geq \dim R+1$. *Moreover, if $R$ is local then one can take $F$ to be the Koszul complex on a sequence of parameters for $R$, and in that case equality holds.*

*Proof.* Let $\mathfrak{m}$ be a maximal ideal of $R$ with $\dim R_\mathfrak{m} = \dim R$. Let $K$ be the Koszul complex on a sequence of parameters for $R_\mathfrak{m}$ and choose by 2.8 a bounded complex $F$ of finitely generated free $R$-modules with $F_\mathfrak{m} \simeq K$. In view of [3, Lemma 2.4(6)] this explains the inequality in the display below. Per [4, Corollary 3.4] and 2.7 the equalities hold as $K$ is the minimal free resolution of $\mathrm{H}_0(K)$,

$$\begin{aligned} \mathrm{level}_R^{\mathsf{G}(R)}\, F &\geq \mathrm{level}_{R_\mathfrak{m}}^{\mathsf{G}(R_\mathfrak{m})}\, K \\ &= \mathrm{level}_{R_\mathfrak{m}}^{\mathsf{G}(R_\mathfrak{m})}\, \mathrm{H}_0(K) \\ &= \mathrm{Gdim}_{R_\mathfrak{m}}\, \mathrm{H}_0(K) + 1 \\ &= \dim R + 1\,. \end{aligned}$$

□

For $d = 0$ the conclusion in the next result is not optimal; see Proposition 3.6.

**Theorem 2.10.** *Let $R$ be a commutative noetherian ring. If there exists an integer $d$ such that* $\mathrm{level}_R^{\mathsf{G}(R)}\, P \leq d+1$ *holds for every perfect $R$-complex $P$, then $R$ is Gorenstein of Krull dimension at most $d$.*

*Proof.* Let $\mathfrak{m}$ be a maximal ideal of $R$; it suffices to show that the local ring $R_\mathfrak{m}$ is Gorenstein of Krull dimension at most $d$. Let $P$ be a perfect $R_\mathfrak{m}$-complex. Per 2.8 there exists a perfect $R$-complex $F$ with $F_\mathfrak{m} \simeq P$ in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R_\mathfrak{m})$. Now, [3, Lemma 2.4(6)] yields the first inequality in the chain

$$\mathrm{level}_{R_\mathfrak{m}}^{\mathsf{G}(R_\mathfrak{m})}\, P = \mathrm{level}_{R_\mathfrak{m}}^{\mathsf{G}(R_\mathfrak{m})}\, F_\mathfrak{m} \leq \mathrm{level}_R^{\mathsf{G}(R)}\, F \leq d+1\,,$$

which implies that $R_{\mathfrak{m}}$ is Gorenstein by Theorem 2.5.

Finally, one has $\dim R_{\mathfrak{m}} \le d$ by Lemma 2.9. □

**Remark 2.11.** Let $M$ be a complex in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ of finite Gorenstein projective dimension. It follows from [7, Proposition 9.1.27] that $M$ fits in a triangle with a perfect $R$-complex and a module from $\mathsf{G}(R)$. This provides some measure of an *a posteriori* explanation of why it suffices to bound the $\mathsf{G}(R)$-level of perfect $R$-complexes to conclude that $R$ is Gorenstein.

## 3. G-levels over Gorenstein rings

**3.1.** For an $R$-complex $M$, set $M^{\oplus} \coloneqq \bigoplus_{n\in\mathbb{Z}} M_n$. This notation comes in handy as $\operatorname{level}_R^{\mathsf{G}(R)} M = \operatorname{level}_R^{\mathsf{G}(R)} M^{\oplus}$ holds for a complex $M$ with zero differential.

The next result follows from work of Aihara, Araya, Iyama, Takahashi, and Yoshiwaki [1, Theorem 4.1]; it can be also proved by an argument dual to the proof of Proposition 4.8.

**Proposition 3.2.** *For every complex $M$ in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ one has*

$$\operatorname{level}_R^{\mathsf{G}(R)} M \le \max\{2, \operatorname{Gdim}_R \mathrm{H}(M)^{\oplus} + 1\}\,.$$

**Remark 3.3.** The upper bound on $\mathsf{G}(R)$-levels given above can be attained: Indeed, $\operatorname{level}_R^{\mathsf{G}(R)} M = \operatorname{Gdim}_R M + 1$ holds for every $R$-module $M$ by [4, Corollary 3.4], whence the bound is attained by every module of positive G-dimension. Thus, one may rewrite the upper bound as

$$\operatorname{level}_R^{\mathsf{G}(R)} M \le \max\{2, \operatorname{level}_R^{\mathsf{G}(R)} \mathrm{H}(M)^{\oplus}\}\,.$$

Moreover, there exists by [4, Example 3.10] an $R$-complex $M$ with $\mathrm{H}(M)^{\oplus}$ in $\mathsf{G}(R)$ but $\operatorname{level}_R^{\mathsf{G}(R)} M = 2$.

The following is an immediate consequence of Proposition 3.2:

**Theorem 3.4.** *Let $R$ be Gorenstein and $M$ a complex in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$. One has*

$$\operatorname{level}_R^{\mathsf{G}(R)} M \le \max\{2, \dim R + 1\}\,.$$

Since $R$ is regular of finite Krull dimension if and only if there exists a uniform upper bound on the $R$-level of all perfect $R$-complexes, Theorem 3.4 tells us that over a nonregular Gorenstein ring there must exist a perfect complex $P$ such that the inequality, $\operatorname{level}_R^{\mathsf{G}(R)} P \le \operatorname{level}_R^{R} P$, which holds as $R$ belongs to $\mathsf{G}(R)$, is strict.

**Proposition 3.5.** *Let $R$ be Gorenstein and not regular and $n \ge 2$ an integer. There exists a perfect $R$-complex $P$ with $\operatorname{level}_R^{R} P = n+1$ and $\operatorname{level}_R^{\mathsf{G}(R)} P \le 2$.*

*Proof.* We build off an argument by Altmann, Grifo, Montaño, Sanders, and Vu [2, Corollary 2.3]. Let $G$ be a module in $\mathsf{G}(R)$ and not projective, i.e. of infinite projective dimension. If $F \to G$ is a free resolution, then the truncated complex

$$P \coloneqq 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0$$

has $R$-level $n+1$ by the proof of [2, Corollary 2.3]. Further, there is a triangle

$$\Sigma^n \mathrm{H}_n(P) \longrightarrow P \longrightarrow \mathrm{H}_0(P) \longrightarrow$$

in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$, so $\operatorname{level}_R^{\mathsf{G}(R)} P \le 2$ holds by [3, Lemma 2.4(2)] as both modules $\mathrm{H}_n(P)$ and $\mathrm{H}_0(P) \cong G$ are Gorenstein projective modules. □

The inequality in Theorem 3.2 begs the question: What happens if the $\mathsf{G}(R)$-levels of complexes in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ are at most 1?

**Proposition 3.6.** *If* $\operatorname{level}^{\mathsf{G}(R)}_R P \le 1$ *holds for every perfect $R$-complex $P$, then $R$ is regular of Krull dimension* 0.

*Proof.* It follows from Theorem 2.10 that $R$ has Krull dimension 0. Let $\mathfrak{m}$ be a maximal ideal in $R$; it suffices to show that $R_\mathfrak{m}$ is regular. Let $P$ be a perfect complex over $R_\mathfrak{m}$; per 2.8 there exists a perfect $R$-complex $F$ with $F_\mathfrak{m} \simeq P$. Now, [3, Lemma 2.4(6)] gives the first inequality below

$$\operatorname{level}^{\mathsf{G}(R_\mathfrak{m})}_{R_\mathfrak{m}} P = \operatorname{level}^{\mathsf{G}(R_\mathfrak{m})}_{R_\mathfrak{m}} F_\mathfrak{m} \le \operatorname{level}^{\mathsf{G}(R)}_R F \le 1.$$

One can now assume that $R$ is local. Let $K$ be the Koszul complex on a minimal generating set for the maximal ideal of $R$ and let $k$ be the residue field of $R$. Since $\operatorname{level}^{\mathsf{G}(R)}_R K = 1$ holds, $K$ is isomorphic in the derived category to a complex $M$ that is a direct sum of shifts of modules from $\mathsf{G}(R)$, and so we have

$$K \simeq M \simeq \mathrm{H}(M) \simeq \mathrm{H}(K) = \bigoplus \Sigma^i \mathrm{H}_i(K)$$

Since $\mathrm{H}_0(K) = k \neq 0$, we have $\mathrm{H}_i(K) = 0$ for all $i \neq 0$ by [2, Proposition 4.7], and so $\operatorname{pd}_R k < \infty$. This implies $R$ is regular. □

**Theorem 3.7.** *Let $R$ be commutative noetherian; the following assertions hold.*

(a) *The next conditions are equivalent.*
  ($i$) $\sup\{\operatorname{level}^{\mathsf{G}(R)}_R M \mid M \in \mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)\} = 2$ *holds.*
  ($ii$) $\sup\{\operatorname{level}^{\mathsf{G}(R)}_R P \mid P \text{ is a perfect } R\text{-complex}\} = 2$ *holds.*
  ($iii$) *$R$ is Gorenstein with* $\dim R = 1$ *or Gorenstein with* $\dim R = 0$ *and not regular.*

(b) *Let $d \ge 2$ be an integer. The following are equivalent.*
  ($i$) $\sup\{\operatorname{level}^{\mathsf{G}(R)}_R M \mid M \in \mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)\} = d+1$ *holds.*
  ($ii$) $\sup\{\operatorname{level}^{\mathsf{G}(R)}_R P \mid P \text{ is a perfect } R\text{-complex}\} = d+1$ *holds.*
  ($iii$) *$R$ is Gorenstein with* $\dim R = d$.

*Proof.* (a): Assume that the equality in ($i$) holds. It follows from Theorem 2.10 that $R$ is Gorenstein of Krull dimension at most 1. If $\dim R$ equals 1, then Lemma 2.9 implies the existence of a perfect $R$-complex of $\mathsf{G}(R)$-level 2. If $\dim R = 0$ holds, then $R$ is per [3, Theorem 5.5] not regular, since there exists a complex in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ of $\mathsf{G}(R)$-level 2 and hence $R$-level at least 2. Now, Proposition 3.6 implies the existence of a perfect $R$-complex of $\mathsf{G}(R)$-level 2. Thus, ($i$) implies both ($ii$) and ($iii$) and, more importantly, the arguments above apply verbatim to show that ($ii$) implies ($iii$). It remains to show that ($iii$) implies ($i$):

If $R$ is Gorenstein with $\dim R \le 1$, then 2 is by Theorem 3.4 an upper bound on the $\mathsf{G}(R)$-level of complexes in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$. If $\dim R = 1$ holds, then Lemma 2.9 implies the existence of a complex in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ of $\mathsf{G}(R)$-level 2. If $\dim R = 0$ holds and $R$ is not regular, then Proposition 3.6 implies the existence of a complex in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ of $\mathsf{G}(R)$-level 2.

(b): We proceed by induction on $d$. Let $d = 2$. If the equality in ($i$) or ($ii$) holds, then it follows from Theorem 2.10 and part (a) that $R$ is Gorenstein of Krull dimension 2. On the other hand, if $R$ is such a ring, then there exists by Lemma

2.9 a perfect $R$-complex of $\mathsf{G}(R)$-level 3, and 3 is by Theorem 3.4 an upper bound on the $\mathsf{G}(R)$-level of complexes in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$.

Now, let $d > 2$. If the equality in $(i)$ or $(ii)$ holds, then it follows from Theorem 2.10 that $R$ is Gorenstein of Krull dimension at most $d$, and by the induction hypothesis the Krull dimension is at least $d$. On the other hand, if $R$ is such a ring, then there exists by Lemma 2.9 a perfect $R$-complex of $\mathsf{G}(R)$-level $d+1$, and $d+1$ is by Theorem 3.4 an upper bound on the $\mathsf{G}(R)$-level of complexes in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$. $\square$

**Example 3.8.** Let $R$ be local with maximal ideal $\mathfrak{m}$, and let $M$ be a complex in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ or a derived $\mathfrak{m}$-complete $R$-complex. Assume that $M$ has finite Gorenstein dimension and set $s \coloneqq \sup \mathrm{H}_*(M)$. If $\Gamma_I \mathrm{H}_s(M) \neq 0$ holds, then we have

$$\begin{aligned} \operatorname{Gdim}_R M &= \operatorname{depth} R - \operatorname{depth} M \\ &\geq \operatorname{depth} R - \operatorname{depth}(I, M) - \dim R/I \\ &= \operatorname{depth} R + s - \dim R/I \,. \end{aligned}$$

The first equality is by 2.7. The inequality holds by work of Iyengar [9, Proposition 5.5(4)] if $M$ is in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ and by work of Christensen and Ferraro [6, Corollary 1.3] if $M$ is derived $\mathfrak{m}$-complete. The last equality is standard when $\Gamma_I \mathrm{H}_s(M) \neq 0$, see [7, Theorem 14.3.16]. Now, per [4, Theorem 3.3] one has:

$$\operatorname{level}_R^{\mathsf{G}(R)} M \geq \operatorname{Gdim}_R M - s + 1.$$

Hence, combining these two inequalities, one gets:

$$\operatorname{level}_R^{\mathsf{G}(R)} M \geq \operatorname{depth} R - \dim R/I + 1.$$

This example gives another proof that the $\mathsf{G}(R)$-level of the Koszul complex on a sequence of parameters in a Cohen–Macaulay local ring $R$ is exactly $\dim R + 1$, cf. Lemma 2.9.

**Example 3.9.** Let $R$ be local and Cohen–Macaulay. If $\underline{x} \coloneqq x_1, \ldots, x_n$ is a generating set for a proper ideal $I$ of $R$, then Example 3.8 yields

$$\operatorname{level}_R^{\mathsf{G}(R)} K(\underline{x}; R) \geq \dim R - \dim R/I + 1.$$

Further, if $\underline{x}$ forms a (partial) sequence of parameters for $R$, then

$$\operatorname{level}_R^{\mathsf{G}(R)} K(\underline{x}; R) = \dim R - \dim R/I + 1 = n + 1$$

holds. Indeed, one has

$$\operatorname{level}_R^{\mathsf{G}(R)} K(\underline{x}; R) \leq \operatorname{level}_R^{R} K(\underline{x}; R) \leq n + 1$$

where the first equality holds as $R$ belongs to $\mathsf{G}(R)$ and the second inequality is from the length of the Koszul complex, see [3, Lemma 2.5.2].

Finally, we remark that the $\mathsf{G}(R)$- and $R$-levels of a Koszul complex can differ.

**Example 3.10.** Let $R$ be local and Gorenstein of positive Krull dimension. Let $M$ be a complex in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ with $s \coloneqq \sup \mathrm{H}_*(M)$. If $\Gamma_{\mathfrak{m}} \mathrm{H}_s(M) \neq 0$, then we have

$$\operatorname{level}_R^{\mathsf{G}(R)} M = \dim R + 1 \,.$$

One inequality holds by Theorem 3.4 and the opposite inequality comes from Example 3.8. Thus, for the Koszul complex $K$ on a set of generators of the maximal ideal of $R$ one has $\operatorname{level}_R^{\mathsf{G}(R)} K = \dim R + 1$ while $\operatorname{level}_R^R M = \operatorname{edim} R + 1$ holds by [2, Theorem 4.2(1)]. Thus if $R$ is not regular, then one has $\operatorname{level}_R^{\mathsf{G}(R)} K < \operatorname{level}_R^R K$.

## 4. Gorenstein injective levels

The next construction is dual to the Adams resolution in the sense of J.D. Christensen [5].

**Construction 4.1.** Let $M$ be an $R$-complex. There is a graded injective $R$-module $I$ and an injective homomorphism $\bar{\iota}\colon \mathrm{H}(M) \hookrightarrow I$ of graded $R$-modules. By the graded-injectivity of $I$, this map lifts to a graded homomorphism $\mathrm{C}(M) \to I$. Considering $I$ as a complex with zero differential, this yields a morphism $\iota\colon M \to I$ of complexes.

Set $\mho_R^0(M) \coloneqq M$ and $\mho_R^1(M) \coloneqq \operatorname{cone}(\iota)$. Applying the construction above recursively, set $\mho_R^{n+1}(M) \coloneqq \mho_R^1(\mho_R^n(M))$ for $n \geq 1$. Taken together the ensuing triangles

$$\mho^n(M) \xrightarrow{\iota^n} I^n \longrightarrow \mho^{n+1}(M) \longrightarrow$$

form an injective Adams resolution for $M$. They induce short exact sequences in homology

$$0 \longrightarrow \mathrm{H}(\mho_R^n(M)) \longrightarrow I^n \longrightarrow \mathrm{H}(\mho^{n+1}(M)) \longrightarrow 0$$

for every $n \geq 0$.

**Lemma 4.2.** *Let $\mathsf{C}$ be a collection of objects in $\mathsf{D}(R)$ that contains all injective $R$-modules. For every complex $M$ in $\mathsf{D}_\mathsf{b}(R)$ and every $n \geq 0$ there is an inequality*

$$\operatorname{level}_R^{\mathsf{C}} M \leq \operatorname{level}_R^{\mathsf{C}} \mho^n(M) + n$$

*Proof.* It suffices to show that one has

$$\operatorname{level}_R^{\mathsf{C}} M \leq \operatorname{level}_R^{\mathsf{C}} \mho^1(M) + 1\,.$$

To this end consider the triangle

$$M \xrightarrow{\iota^0} I^0 \longrightarrow \mho^1(M) \longrightarrow$$

and notice that $I^0$ is a bounded complex of injective modules with zero differential. Now [3, Lemma 2.4(2)] yields the desired inequality. □

**Lemma 4.3.** *Let $M \in \mathsf{D}(R)$. For every integer $n \geq 1$ there is an exact sequence*

$$0 \longrightarrow \mathrm{H}(M)^{\oplus} \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^{n-1} \longrightarrow \mathrm{H}(\mho^n(M))^{\oplus} \longrightarrow 0$$

*where each $E^i$ is an injective $R$-module.*

*Proof.* The claim holds since for every $i \geq 0$ there is an exact sequence of $R$-modules

$$0 \longrightarrow \mathrm{H}(\mho^n(M))^{\oplus} \longrightarrow (I^n)^{\oplus} \longrightarrow \mathrm{H}(\mho^{n+1}(M))^{\oplus} \longrightarrow 0$$

from Construction 4.1. □

**4.4.** In the next several results, we make use of the following exact sequences from [7, Proposition 2.2.12]:

$$0 \longrightarrow \mathrm{Z}_i(M) \longrightarrow M_i \longrightarrow \mathrm{B}_{i-1}(M) \longrightarrow 0 \tag{1}$$

$$0 \longrightarrow \mathrm{B}_i(M) \longrightarrow M_i \longrightarrow \mathrm{C}_i(M) \longrightarrow 0 \tag{2}$$

$$0 \longrightarrow \mathrm{H}_i(M) \longrightarrow \mathrm{C}_i(M) \longrightarrow \mathrm{B}_{i-1}(M) \longrightarrow 0 \tag{3}$$

$$0 \longrightarrow \mathrm{B}_i(M) \longrightarrow \mathrm{Z}_i(M) \longrightarrow \mathrm{H}_i(M) \longrightarrow 0 \tag{4}$$

**Lemma 4.5.** *Suppose $M$ is a left bounded $R$-complex such that $M_i$ and $\mathrm{H}_i(M)$ have finite Gorenstein injective dimension for all $i$. Then $\mathrm{B}_i(M)$, $\mathrm{Z}_i(M)$, and $\mathrm{C}_i(M)$ have finite Gorenstein injective dimension for all $i$ as well.*

*Proof.* If $\mathrm{Z}_i(M)$ has finite Gorenstein injective dimension, then the modules $\mathrm{B}_{i-1}(M)$ and $\mathrm{C}_i(M)$ have finite Gorenstein injective dimension by 4.4(1) and 4.4(3), and then $\mathrm{Z}_{i-1}(M)$ has finite Gorenstein injective dimension by 4.4(4). The assertion thus follows by descending induction, as $\mathrm{Z}_i(M) = 0$ for $i \gg 0$. $\square$

**Proposition 4.6.** *Let $M$ be a complex in $\mathsf{D_b}(R)$. If the module $\mathrm{H}(M)^{\oplus}$ has finite Gorenstein injective dimension, then $M$ has finite Gorenstein injective dimension.*

*Proof.* Let $I$ be a semi-injective resolution of $M$. Since every module $I_i$ has finite Gorenstein injective dimension, it follows from Lemma 4.5 that $\mathrm{Z}_i(I)$ has finite Gorenstein injective dimension for all $i$. Since $\mathrm{H}_i(I) = 0$ holds for $i \ll 0$, one has $\operatorname{Gid}_R M \le \operatorname{Gid}_R \mathrm{Z}_i(M) - i$ for $i \ll 0$. $\square$

In the following $\mathsf{GI}(R)$ denotes the collection of Gorenstein injective $R$-modules.

**Lemma 4.7.** *Let $M$ be a left bounded complex of modules from $\mathsf{GI}(R)$. If one has $\operatorname{Gid}_R \mathrm{H}_i(M) \le 1$ for all $i$, then the modules $\mathrm{B}_i(M)$ and $\mathrm{C}_i(M)$ belong to $\mathsf{GI}(R)$ for all $i$.*

*Proof.* If $\mathrm{B}_i(M)$ is in $\mathsf{GI}(R)$, then $\mathrm{C}_i(M)$ is in $\mathsf{GI}(R)$ by 4.4(2), and per the assumption on $\mathrm{H}_i(M)$ it follows from 4.4(4) that $\mathrm{B}_{i-1}(M)$ is in $\mathsf{GI}(R)$. The assertion thus follows by descending induction as one has $\mathrm{B}_i(M) = 0$ for $i \gg 0$. $\square$

The next theorem is dual to Proposition 3.2; it does not follow from work in [1] which only deals with resolving categories.

**Proposition 4.8.** *For every complex $M$ in $\mathsf{D_b}(R)$ one has:*

$$\operatorname{level}_R^{\mathsf{GI}(R)} M \le \max\{2, \operatorname{Gid}_R \mathrm{H}(M)^{\oplus} + 1\}$$

*Proof.* We may assume that $M$ is nonzero and that the Gorenstein injective dimension of $\mathrm{H}(M)^{\oplus}$ is finite; otherwise there is nothing to prove. Now the Gorenstein injective dimension of $M$ is finite by Proposition 4.6. Thus, we may assume that $M$ is a bounded complex of Gorenstein injective modules.

First, notice that if $\mathrm{B}(M)^{\oplus}$ and $\mathrm{C}(M)^{\oplus}$ belong to $\mathsf{GI}(R)$, then [3, Lemma 2.4(2)] applied to the exact sequence

$$0 \longrightarrow \mathrm{B}(M) \longrightarrow M \longrightarrow \Sigma\,\mathrm{C}(M) \longrightarrow 0$$

from 4.4(2) yields

$$\begin{aligned}\operatorname{level}_R^{\mathsf{GI}(R)} M &\le \operatorname{level}_R^{\mathsf{GI}(R)} \mathrm{B}(M) + \operatorname{level}_R^{\mathsf{GI}(R)} \mathrm{C}(M)\\ &= \operatorname{level}_R^{\mathsf{GI}(R)} \mathrm{B}(M)^{\oplus} + \operatorname{level}_R^{\mathsf{GI}(R)} \mathrm{C}(M)^{\oplus}\\ &\le 2\,,\end{aligned}$$

where the equality holds as the complexes $\mathrm{B}(M)$ and $\mathrm{C}(M)$ have zero differentials. In particular, the asserted inequality holds by Lemma 4.7 in case $\operatorname{Gid}_R \mathrm{H}(M)^{\oplus} \le 1$.

For $n := \operatorname{Gid}_R \mathrm{H}(M)^{\oplus} > 1$ we see that $\operatorname{Gid}_R \mathrm{H}(\mho_R^{n-1}(M))^{\oplus} = 1$ holds by Lemma 4.3, so the modules $\mathrm{B}(\mho_R^{n-1}(M))^{\oplus}$ and $\mathrm{C}(\mho_R^{n-1}(M))^{\oplus}$ belong to $\mathsf{GI}(R)$ by Lemma 4.7. By the argument above, one now has $\operatorname{level}_R^{\mathsf{GI}(R)} \mho_R^{n-1}(M) \le 2$, so Lemma 4.2 yields $\operatorname{level}_R^{\mathsf{GI}(R)} M \le n + 1$. $\square$

**Remark 4.9.** The upper bound on $\mathsf{GI}(R)$-levels provided in Proposition 4.8 can be attained: Indeed, for an $R$-module $M$ one has $\operatorname{level}_R^{\mathsf{GI}(R)} M = \operatorname{Gid}_R M + 1$ by [3, Lemma 2.5.2] and work of Mifune [13, Corollary 4.3]. Thus the bound is attained by every $R$-module of positive Gorenstein injective dimension, and one can thus rewrite the bound as

$$\operatorname{level}_R^{\mathsf{GI}(R)} M \le \max\{2, \operatorname{level}_R^{\mathsf{GI}(R)} \mathrm{H}(M)^{\oplus}\}\,.$$

Next we show that a complex $M$ with $\mathrm{H}(M)^{\oplus}$ in $\mathsf{GI}(R)$ may have $\mathsf{GI}(R)$-level 2.

**Example 4.10.** Let $k$ be a field and consider the local ring $R \coloneqq k[x]/(x^2)$. Let $K$ be the Koszul complex on $x$; as $R$ is artinian and Gorenstein, $\mathrm{H}(K)^{\oplus}$ is a Gorenstein injective $R$-module. By Proposition 4.8 one has $\operatorname{level}_R^{\mathsf{GI}(R)} K \le 2$. If the $\mathsf{GI}(R)$-level of $K$ were 1, then $K$ would be isomorphic to $\mathrm{H}(K)$ in the derived category, which as shown in [4, Example 3.10] is absurd. Thus $\operatorname{level}_R^{\mathsf{GI}(R)} K = 2$ holds.

The following is an immediate consequence of Proposition 4.8:

**Theorem 4.11.** *Let $R$ be Gorenstein and $M$ a complex in $\mathsf{D_b}(R)$. One has*

$$\operatorname{level}_R^{\mathsf{GI}(R)} M \le \max\{2, \dim R + 1\}\,.$$

As another application of Proposition 4.8 we prove a formula, reminiscent of the Bass Formula, for Gorenstein injective level. The assumption that $R$ has positive depth is necessary, for $R$ and $K$ as in Example 4.10 one has $\operatorname{level}_R^{\mathsf{GI}(R)} K = 2$.

**Theorem 4.12.** *Let $R$ be a commutative noetherian local ring of positive depth and $M$ a complex in $\mathsf{D^f_b}(R)$. If $\operatorname{Gid}_R \mathrm{H}(M)^{\oplus}$ is finite, then one has*

$$\operatorname{level}_R^{\mathsf{GI}(R)} M = \operatorname{depth} R + 1$$

*Proof.* By Theorem 4.8, we have $\operatorname{level}_R^{\mathsf{GI}(R)} M \le \max\{2, \operatorname{Gid}_R \mathrm{H}(M)^{\oplus} + 1\}$. Since $\mathrm{H}(M)^{\oplus}$ is a finitely generated $R$-module, the Bass Formula for Gorenstein injective dimension, see [7, Corollary 19.2.14], yields $\operatorname{Gid}_R \mathrm{H}(M)^{\oplus} = \operatorname{depth} R$. By assumption the depth of $R$ is positive, so one now has $\operatorname{level}_R^{\mathsf{GI}(R)} M \le \operatorname{depth} R + 1$.

For the opposite inequality, note first that $\operatorname{Gid}_R M$ is finite by Proposition 4.6. Per [13, Corollary 4.3] one has

$$\operatorname{level}_R^{\mathsf{GI}(R)} M \ge \operatorname{Gid}_R M + \inf M + 1\,,$$

and $\operatorname{Gid}_R M + \inf M = \operatorname{depth} R$ holds by [7, Theorem 19.2.40]. □

We close out this section by remarking that that there is a Bass formula for levels with respect to the class of injective $R$-modules. Indeed, the proof of [3, Theorem 5.5] dualizes to give an upper bound on $\mathsf{I}(R)$-levels, where $\mathsf{I}(R)$ denotes the class of injective $R$-modules.

**Proposition 4.13.** *For every complex $M$ in $\mathsf{D_b}(R)$ one has:*

$$\operatorname{level}_R^{\mathsf{I}(R)} M \le \operatorname{id}_R \mathrm{H}(M)^{\oplus} + 1\,.$$

**Theorem 4.14.** *Let $R$ be a noetherian local ring and $M$ a complex in $\mathsf{D^f_b}(R)$. If $\operatorname{id}_R \mathrm{H}(M)^{\oplus}$ is finite, then one has*

$$\operatorname{level}_R^{\mathsf{I}(R)} M = \operatorname{depth} R + 1\,.$$

*Proof.* One has $\operatorname{level}_R^{\mathsf{I}(R)} M \le \operatorname{depth} R + 1$ by Proposition 4.13 and the classic Bass formula; the opposite inequality holds by [13, Corollary 4.3]. □

## 5. Flat and Gorenstein flat levels

Like Proposition 3.2 the next result follows from [1, Theorem 4.1].

**Proposition 5.1.** *Let $\mathsf{C}$ be the class of flat $R$-modules, Gorenstein projective $R$-modules, or Gorenstein flat $R$-modules. For every complex $M$ in $\mathsf{D_b}(R)$ one has*

$$\operatorname{level}_R^{\mathsf{C}} M \leq \max\{2, \mathsf{C}\text{-}\dim_R \mathrm{H}(M)^{\oplus} + 1\}$$

*where $\mathsf{C}$-$\dim_R$ means flat dimension, Gorenstein projective dimension, or Gorenstein flat dimension, respectively.*

Like the bounds in Propositions 3.2 and 4.8 the ones provided above can be attained. First consider the case where $\mathsf{C}$ is the class $\mathsf{GP}(R)$ of Gorenstein projective $R$-modules. For every $R$-module $M$ one has $\operatorname{level}_R^{\mathsf{GP}(R)} M = \operatorname{Gpd}_R M + 1$ by [4, Corollary 3.4]. Further, for $R$ and $K$ as in Example 4.10, the argument in [4, Example 3.10] shows that while $\mathrm{H}(K)^{\oplus}$ is in $\mathsf{GP}(R)$ one has $\operatorname{level}_R^{\mathsf{GP}(R)} K = 2$.

**Lemma 5.2.** *Let $\mathsf{C}$ be the class of flat $R$-modules or Gorenstein flat $R$-modules. For every $R$-module $M$ one has*

$$\operatorname{level}_R^{\mathsf{C}} M = \mathsf{C}\text{-}\dim_R M + 1\,,$$

*where $\mathsf{C}$-$\dim_R$ means flat dimension dimension or Gorenstein flat dimension.*

*Proof.* First we consider the case where $\mathsf{C}$ is the class $\mathsf{F}(R)$ of flat $R$-modules. Note that the inequality $\operatorname{level}_R^{\mathsf{F}(R)} M \leq \operatorname{fd}_R M + 1$ holds by [3, Lemma 2.5.2]. For the other inequality, let $E$ be a faithfully injective $R$-module. By [3, Lemma 2.4(6)] there is an inequality,

$$\operatorname{level}_R^{\operatorname{Hom}_R(\mathsf{F}(R),E)} \operatorname{Hom}_R(M,E) \leq \operatorname{level}_R^{\mathsf{F}(R)} M\,.$$

For every flat $R$-module $F$, the module $\operatorname{Hom}_R(F,E)$ is injective by flat–injective duality. Thus one has

$$\operatorname{level}_R^{\mathsf{I}(R)} \operatorname{Hom}_R(M,E) \leq \operatorname{level}_R^{\operatorname{Hom}_R(\mathsf{F}(R),E)} \operatorname{Hom}_R(M,E)\,.$$

As $\operatorname{level}_R^{\mathsf{I}(R)} \operatorname{Hom}_R(M,E) = \operatorname{id}_R \operatorname{Hom}_R(M,E) + 1$ holds by Remark 4.9 and one has $\operatorname{id}_R \operatorname{Hom}_R(M,E) = \operatorname{fd}_R M$ by flat-injective duality, the asserted inequality follows.

The same argument applies when $\mathsf{C}$ is the class of Gorenstein flat $R$-modules; only one refers to [7, Proposition 9.3.20] in place of flat–injective duality. ☐

**Remark 5.3.** In the flat and Gorenstein flat cases, the upper bound established in Proposition 5.1 can be attained, as $\operatorname{level}_R^{\mathsf{C}} M = \mathsf{C}\text{-}\dim_R M + 1$ holds for every $R$-module $M$ by Lemma 5.2 .

Next we construct a complex $M$ with $\operatorname{level}_R^{\mathsf{F}(R)} M = 2 = \operatorname{level}_R^{\mathsf{GF}(R)} M$ though the module $\mathrm{H}(M)^{\oplus}$ is flat and, hence, Gorenstein flat.

**Example 5.4.** Set $R := \mathbb{R}[x,y,z]$ and let $Q$ be the field of fractions of $R$. As $R$ is regular, the projective dimension of an $R$-module is at most 3, and every Gorenstein flat $R$-module is flat; in symbols $\mathsf{GF}(R) = \mathsf{F}(R)$. By a result of Osofsky [14], the projective dimension of the flat $R$-module $Q$ is at least two. Let $P$ be a free resolution of $Q$, and consider the hard truncation of $P$ at degree 1, denoted $P_{\leq 1}$. One has $\mathrm{H}_0(P_{\leq 1}) \cong Q$ and $\mathrm{H}_1(P_{\leq 1}) \cong \Omega_R^2(Q)$, the second syzygy of $Q$ which is also a flat $R$-module. Thus, $\mathrm{H}(P_{\leq 1})^{\oplus}$ is flat, whence Proposition 5.1 yields $\operatorname{level}_R^{\mathsf{F}(R)} P_{\leq 1} \leq 2$; we claim that equality holds. Assume towards a contradiction that $\operatorname{level}_R^{\mathsf{F}(R)} P_{\leq 1} = 1$ holds. In the derived category, $P_{\leq 1}$ is now isomorphic to its homology. As $P_{\leq 1}$ is semi-projective, there is a quasi-isomorphism $\pi$ from $P_{\leq 1}$ to its homology:

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0} & & \\ 0 & \longrightarrow & \Omega_R^2(Q) & \xrightarrow{0} & Q & \longrightarrow & 0 \end{array}$$

As $\pi$ is a quasi-isomorphism, its mapping cone is acyclic. Thus, the sequence

$$0 \longrightarrow P_1 \longrightarrow P_0 \oplus \Omega_R^2(Q) \longrightarrow Q \longrightarrow 0$$

is exact. One has $\operatorname{pd}_R P_0 \oplus \Omega_R^2(Q) = \operatorname{pd}_R Q - 2$ as $\Omega_R^2(Q)$ is a second syzygy of $Q$. Since the projective dimension of $Q$ is at most 3, the projective dimension of $P_0 \oplus \Omega_R^2(Q)$ is at most 1. However, from the exact sequence above one gets

$$\begin{aligned} \operatorname{pd}_R Q &\leq \max\{\operatorname{pd}_R P_1 + 1, \operatorname{pd}_R P_0 \oplus \Omega_R^2(Q)\} \\ &\leq \max\{1, 1\} \\ &\leq 1 \end{aligned}$$

by [7, Corollary 8.1.9]. This is a contradiction, and so $\operatorname{level}_R^{\mathsf{F}(R)} Q = 2$.

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Texas Tech University, TX 79409, U.S.A.
*Email address*: lars.w.christensen@ttu.edu
*URL*: https://www.math.ttu.edu/~lchriste

University of Utah, UT 84112, U.S.A.
*Email address*: kekkou@math.utah.edu
*URL*: https://sites.google.com/view/antonia-kekkou/

Auburn University, AL 36849, U.S.A.
*Email address*: jll0107@auburn.edu
*URL*: https://jlyle42.github.io/justinlyle

University of Nebraska, NE 68588, U.S.A.
*Email address*: znason2@huskers.unl.edu
*URL*: https://zach-nason.github.io/

Texas Tech University, TX 79409, U.S.A.
*Email address*: ansotole@ttu.edu
*URL*: https://sites.google.com/view/andrewjsotolevins